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Discussion Paper No. 16-1, May 2016

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A revenue-enhancing effect of a buyout price in multi-unit uniform-price auctions

Toshihiro Tsuchihashi*

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Abstract

A demand reduction is prominent in multi-unit auctions. Buyers lower a bid on a lesser-valued unit and may cause low seller revenue. We focus on a simple situation where two buyers with two-unit demand have a buyout option in multi-unit uniform-price auctions and they each obtain one unit at a very low price if at least one buyer does not exercise a buyout option. By using a simple model, we show that a buyout price improves seller revenue. The revenue-enhancing effect of a buyout price arises from a trade-off faced to buyers between the benefit from obtaining two units at a given buyout price and the cost of losing a chance to win one unit at a low price.

1 Introduction

We frequently observe many items are auctioned for sale at a time in the actual world. These auctions are called multi-unit auctions. The examples of multi-unit auctions include Treasury bill auctions, spectrum auctions, and online auctions such as eBay and Yahoo.

As Vickrey (1961) already pointed in his seminal work, the results obtained in single-unit auctions do not generally apply to multi-unit auctions. Although second-price sealed-bid auctions in the single-unit environment are naturally extended to sealed-bid uniform-price auctions in the multi-unit environment, bidders do not have a dominant strategy in the latter. Generally, bidding sincerely their actual valuations on all the units does not even constitute a symmetric equilibrium.

Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998) study uniform-price sealed-bid auctions with independent private values in which buyers each demand two units of the item and value the second unit lower than the first unit. Noussair (1995) show that an equilibrium bid on one unit is irrelevant to the valuation of another unit. Especially, in a symmetric undominated equilibrium, buyers sincerely bid on the first unit, i.e., they submit the actual valuation. A sincere bid on the first unit is intuitive because the first-unit bid has no impact on the payment for the first unit, similarly with a dominant strategy in second-price sealed-bid auctions.

On the other hand, a buyer has an incentive to lower a bid on the second unit because a buyer pays his second-unit bid for the first unit with a positive probability. A buyer then faces a trade-off between

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the benefit from lowering an expected payment and the cost of decreasing a winning probability. Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998) show that the benefit dominates the cost; that is, in a symmetric undominated equilibrium, a second-unit bid is chosen below the actual valuation.

This bidding strategy, referred to as a demand reduction, is optimal in more general environments involving many buyers who demand more than two units. Ausubel and Cramton (1996) and Ausubel, et al. (2014) consider uniform-price sealed-bid auctions in which infinitely divisible items are auctioned and show that a buyer optimally lowers bids on lesser-valued units if at least one buyer has downward-sloping demand.

A demand reduction yields inefficient allocations and lowers seller revenue. In the extreme, all units are sold for a zero price. Engelbrecht-Wiggans and Kahn (1998) provide a necessary condition for a buyer with two-unit demand to submit zero as a second-unit bid. In this single-unit bid equilibrium, seller revenue falls to zero.

Experimental studies confirm a demand reduction in uniform-price sealed-bid auctions in the context of comparing multi-unit auction formats.[1] Kagel and Levin (2001) conduct a laboratory experiment of multi-unit auctions in which a subject with two-unit demand competes in bid with a computer with single-unit. The experimenters program the computer to follow a dominant strategy. The environment theoretically leads to a demand reduction because one buyer (i.e., the computer) has downward-sloping demand. As theory predicts, they find that a buyer lowers the second-unit bid in uniform-price sealed-bid auctions as compared with dynamic Vickrey (or Ausubel) auctions in which buyers optimally reveal their actual demand. Similarly, List and Lucking-Reiley (2000) find a significant ratio of a demand reduction and a zero bid on the second unit in uniform-price sealed-bid auctions as compared with sealed-bid Vickrey auctions in their field experiment where two sports trading cards are sold against two buyers with two-unit demand. The similar results are obtained in Porter and Vragov (2006). Engelbrecht-Wiggans, et al. (2006) obtain the similar result in the experiment where more bidders are engaged in. Moreover, these studies in common report overbidding on the first unit.

A demand reduction matters in the real world multi-unit auctions. Ausubel and Cramton (1996) discuss strategic similarity between uniform-price sealed-bid auctions and the Federal Communications Commission (FCC) spectrum auctions and suggest that the outcome of the FCC spectrum auctions show a demand reduction.[2]

It seems valuable to consider ways to deter a demand reduction in uniform-price sealed-bid auctions.

[1]The important exception is Alsemgeest, et al. (1998), who do not observe a demand reduction in uniform-price sealed-bid auctions but in multi-unit English clock auctions.

[2]Grimm, et al. (2003) report a demand reduction in the second-generation (GSM) spectrum auction in Germany which is a simultaneous ascending-bid multi-unit auction. A demand reduction theoretically constitutes a symmetric equilibrium in simultaneous ascending auctions (Engelbrecht-Wiggans and Kahn, 2005).

In this paper, we suggest that introducing a buyout option can prevent buyers from a demand reduction and hence improves seller revenue. We use a simple two-stage model in which two items are sold at auction to two buyers with two-unit demand. Buyers simultaneously decide whether to exercise a buyout option in the first stage, and then compete in bid in the second stage, a uniform-price sealed-bid auction, unless the items are sold in the first stage. Moreover, we focus on an equilibrium in which buyers submit a single-unit bid in the second stage and hence each obtain one unit at a zero price. Such a buyout option shows a trade-off faced to buyers between the benefit from obtaining two units at a given buyout price and the cost of losing a chance to win one unit at a zero price.

This paper therefore has a contribution to the literature of buyout prices as well. To the best of my knowledge, no paper studies a buyout price in multi-unit auctions.[3] In the literature, theoretical studies suggest that a buyout price enhances seller revenue if buyers are risk averse (Budish and Takeyama, 2001; Reynolds and Wooders, 2009) or impatient in time (Mathews, 2004; Mathews and Katzman, 2006).[4] This paper provides other explanation to the literature that a buyout price increases seller revenue.

The organization of this paper is as follows. Section 2 models uniform-price auctions where buyers have a buyout option as a two-stage game. Section 3 considers a buyer's optimal decision on bidding and exercising a buyout option by backward induction and derives a symmetric undominated equilibrium. We see a revenue-enhancing effect of a buyout option in the equilibrium. Section 4 provides a flat-demand example. Section 5 discusses limitations of the analysis and then directs the future research.

2 The model

We model a uniform-price auction with an exogenous buyout price. We focus on a simple situation where two identical items are demanded by two risk-neutral buyers with two-unit demand. A buyer has a quasi-linear payoff function. That is, given payment p per unit, a payoff of a buyer who values kth unit at x_k is given by x_1+x_2-2p if he obtains two units whereas x_1-p if he obtains one unit. Each buyer's valuations (x_1, x_2) are independently and identically drawn from $X = \{(x_1, x_2) \in [0, \bar{x}]^2 | x_1 \geq x_2\}$ according to a joint distribution function $F(x_1, x_2)$. Moreover, we let $f(x_1, x_2) = \frac{\partial^2 F}{\partial x_1 \partial x_2}(x_1, x_2)$ be the corresponding density function and assume $f(x_1, x_2) > 0$ for any $(x_1, x_2) \in X$ (i.e., $F(x_1, x_2)$ has a full support).

[3]There exists a paper that study a buyout price in terms of multi-unit *demand*. Kirkegaard and Overgaard (2008) consider a situation where buyers with two-unit demand sequentially participate in two second-price sealed-bid single-unit auctions. In their model, the first seller is solely allowed to post a buyout price. They show that introducing a buyout option improves the first seller revenue whereas reduces revenue of the second seller.

[4]The literature suggests other theoretical explanations about a rational usage of a buyout price. Tsuchihashi (2015) provides a comprehensive survey of the buyout price literature.

The auction consists of two stages: a *buyout stage* and a *bid stage*. In the buyout stage, given buyout price b, two buyers simultaneously choose quantity q of items they want to buy at the buyout price, $q \in \{0, 1, 2\}$. Let X_q denote a set of valuations that a buyer chooses quantity q, $X = \cup_{q=0}^{2} X_q$. If both of two buyers choose a positive quantity and the total quantity exceeds two, items are randomly assigned to buyers in proportion as their quantities. If the total quantity equals two, buyer(s) purchase items at the buyout price. In these cases, the game ends with the current stage. If one buyer solely exercises a buyout option for one unit, the buyer obtains one unit by paying the buyout price and then exists the auction. The other buyer remains and the game proceeds to the bid stage. On the other hand, both buyers choose zero quantity (i.e., they do not exercise a buyout option), then the game proceeds to the bid stage.

In the bid stage, if both two buyers remain, they participate a uniform-price sealed-bid auction and simultaneously submit two bids (p_1, p_2) where p_k represents a bid on the kth unit. The selling price is determined at the highest rejected bid. On the other hand, if one buyer solely appears the bid stage, he is awarded one unit at a zero price with certainty.[5] Note that this situation happens when either one of the two buyers buys one unit in the buyout stage. By using this model, we consider a buyer's optimal decision on exercising a buyout option and bidding in a symmetric undominated Bayesian equilibrium.

3 Optimal bid

As always, we derive an equilibrium by backward induction and thus first consider a bidding behavior in the bid stage. Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998) find that a demand reduction generally occurs in uniform-price auctions. Specifically, it is optimal for a buyer to bid sincerely on the first unit, i.e., $p_1 = x_1$, in a symmetric undominated equilibrium. However, a buyer optimally lowers a bid on the second unit, i.e., $p_2 \leq x_2$. Engelbrecht-Wiggans and Kahn (1998) show that, as the extreme case, there exists a *single-unit bid* equilibrium in which buyers submit a zero bid on the second unit, i.e., $p_2 = 0$. In this equilibrium, seller revenue falls to zero because both two units are sold for a zero price.

Since this paper is motivated by low revenue observed in multi-unit auctions, we consider the situation yielding zero revenue as the benchmark. Therefore, in what follows, we restrict our attention to an equilibrium in which a buyer submits the single-unit bid in the bid stage. With the following assumption on the distribution function F, we can show that there exists a symmetric undominated

[5]This situation leads the bid stage to a second-price sealed-bid auction in which a single buyer is involved. The non-competitive situation is sometimes observed in online auctions. A seller sets a starting price at the lowest level allowed in the system (e.g., one yen in Yahoo auctions in Japan), and an item is indeed sold at the starting price (perhaps against the seller's intention).

equilibrium in which a buyer submits the single-unit bid.

Assumption 1. We define $X_\mu = \{(x_1, x_2) \in X | x_2 \leq \mu(x_1)\} \subset X$ and let $F_1(x_1|X_\mu)$ denote the conditional marginal distribution. For all $X_\mu \subset X$, F satisfies

$$(\bar{x} - p)\frac{f_1(p|X_\mu)}{1 - F_1(p|X_\mu)} \leq 1, \tag{1}$$

for all $p \leq \bar{x}$.

The equation (1) corresponds to (4.4) in Engelbrecht-Wiggans and Kahn (1998) and they show that for $X_\mu = X$, it is a weakly dominant strategy for a buyer with any valuations to submit the single-unit bid. Lowering a second-unit bid decreases a probability of winning two units whereas reduces the expected payment on the second unit. With the assumption 1, the latter dominates the former for a buyer with any valuations as the lemma shows below. However, the assumption 1 is more strict than Engelbrecht-Wiggans and Kahn's sufficient condition because (1) must hold for subsets of X as well.

Lemma 1. Suppose the assumption 1 holds. It is optimal for a buyer with any valuations (x_1, x_2) to submit a single-unit bid $(p_1, p_2) = (x_1, 0)$ against the same behavior of his opponent.

Proof. Without a loss of generality, we focus on the buyer 1 with valuation (x_1, x_2). As Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998) show, the buyers sincerely bid on the first unit, i.e., $p_1 = x_1$. Thus, we let (x_1, p) be the buyer 1's bid. Suppose that the buyer 2 with valuations (y_1, y_2) submits the single-unit bid $(p_1, p_2) = (y_1, 0)$. Since the two buyers appear in the bid stage if and only if they choose $q = 0$ in the buyout stage, the buyer 2's valuations are belonging to X_0. Let $F_1(y_1|X_0)$ denote the distribution of the buyer 2's first-unit valuation conditional on choosing $q = 0$.

If $p < y_1$, the buyer 1 wins two unit and pays the highest rejected bid y_1 per unit. Otherwise, the buyer 1 obtains one unit and pays his own second-unit bid p. The conditional expected payoff of the buyer 1 to submit $(p_1, p_2) = (x_1, p)$ is given by

$$\begin{aligned}
\pi(p, x_1, x_2) &= \int_0^p (x_1 + x_2 - 2y_1)dF_1(y_1|X_0) + \int_p^{\bar{x}} (x_1 - p)dF_1(y_1|X_0) \\
&= (x_1 + x_2)F_1(p|X_0) - 2\int_0^p y_1 dF_1(y_1|X_0) + (x_1 - p)[1 - F_1(p|X_0)] \\
&= (x_2 - p)F_1(p|X_0) + 2\int_0^p F_1(y_1|X_0)dy_1 + (x_1 - p).
\end{aligned}$$

By differentiating the expected payoff with respect to p, we obtain

$$\frac{\partial \pi}{\partial p}(p, x_1, x_2) = -[1 - F_1(p|X_0)] + (x_2 - p)f_1(p|X_0).$$

Given X_0, if the equation is non-positive for any $x_2 \in [0, \bar{x}]$ and $p \leq x_2$, the buyer 1 optimally chooses $p = 0$. That is,

$$-[1 - F_1(p|X_0)] + (\bar{x} - p) f_1(p|X_0) \leq 0.$$

Since a buyer has a stronger incentive to exercise a buyout option if he has higher valuations, X_0 must locate in a "left-lower" region in a (x_1, x_2)-plane. Thus, the assumption 1 ensures that the above inequality holds for any $X_0 \subset X$. □

4 Buyout option

As the lemma 1 shows, as long as the assumption 1 holds, it is optimal for a buyer with any valuations to submit the single-unit bid against the same behavior of his opponent. Thus, they each obtain one unit at a zero price whenever the auction proceeds to the bid stage. On the other hand, a buyer has two major reasons to exercise a buyout option in the buyout stage. First, a buyer has to exercise an option if he wishes to obtain two units. Second, a buyer may obtain nothing if his opponent exercises an option. These reasons show the basic trade-off between exercising a buyout option or not faced to a buyer. In this section, we consider the optimal decision of the buyer 1 on how many units to buy in the buyout stage. When the buyer 1 exercises a buyout option and chooses $q > 0$, whether he can successfully buy the item(s) depends upon the buyer 2's decision. With a little abuse of notation, we denote by $F(X_q)$ the probability that a buyer has valuations $(x_1, x_2) \in X_q$. We assume that $F(X_2) + F(X_1) + F(X_0) = 1$ holds.

First, suppose that the buyer 1 chooses $q = 2$. There are three cases regarding the buyer 2's choice. If the buyer 2 also chooses $q = 2$, the buyer 1 successfully obtains two units with probability 1/6, one unit with probability 4/6, and nothing with probability 1/6. Similarly, if the buyer 2 chooses $q = 1$, the buyer 1 successfully obtains two units with probability 1/3 and one unit with probability 2/3. On the other hand, if the buyer 2 chooses $q = 0$, the buyer 1 is awarded two units with certainty. Thus, given buyout price b, the expected payoff $\Pi_2(x_1, x_2, b)$ of the buyer 1 to choose $q = 2$ is given by

$$\begin{aligned}\Pi_2(x_1, x_2, b) = F(X_2)[&\frac{1}{6}(x_1 + x_2 - 2b) + \frac{4}{6}(x_1 - b)] \\ &+ F(X_1)[\frac{1}{3}(x_1 + x_2 - 2b) + \frac{2}{3}(x_1 - b)] + F(X_0)(x_1 + x_2 - 2b) \\ = [&\frac{1}{6}F(X_2) + \frac{1}{3}F(X_1) + F(X_0)](x_1 + x_2 - 2b) + \frac{2}{3}[F(X_2) + F(X_1)](x_1 - b) \\ = [&1 - \frac{1}{6}F(X_2)](x_1 - b) + [\frac{1}{6}F(X_2) + \frac{1}{3}F(X_1) + F(X_0)](x_2 - b).\end{aligned}$$

Note that the choice of $q = 2$ makes the game end with the buyout stage with certainty.

Second, suppose that the buyer 1 chooses $q = 1$. If the buyer 2 chooses $q = 2$, the buyer 1 is successfully awarded one unit with probability 2/3 whereas he obtains nothing with probability 1/3.

On the other hand, if the buyer 2 chooses $q < 2$, the buyer 1 obtains one unit with certainty. Thus, the expected payoff $\Pi_1(x_1, x_2, b)$ of the buyer 1 to choose $q = 1$ is given by

$$\Pi_1(x_1, x_2, b) = [\frac{2}{3}F(X_2) + F(X_1) + F(X_0)](x_1 - b)$$
$$= [1 - \frac{1}{3}F(X_2)](x_1 - b).$$

Third, suppose that the buyer 1 does not exercise a buyout option and chooses $q = 0$. If the buyer 2 chooses $q = 2$, the buyer 1 obtains nothing. On the other hand, if the buyer 2 chooses $q < 2$, the auction proceeds to the bid stage and then the buyer 1 obtains one unit at a zero price with certainty. Thus, the expected payoff $\Pi_0(x_1, x_2, b)$ of the buyer 1 choosing $q = 0$ is given by

$$\Pi_0(x_1, x_2, b) = [F(X_1) + F(X_0)](x_1 - 0)$$
$$= [1 - F(X_2)]x_1.$$

The buyer 1 should optimally choose q in order to maximize his expected payoff derived above. We separately consider three cases regarding the buyer 1's valuations: (i) $x_2 \leq x_1 < b$, (ii) $x_2 < b \leq x_1$, and (iii) $b < x_2 \leq x_1$. The buyer 1 never chooses $q > 0$ in the case (i) whereas he never chooses $q = 2$ in the case (ii). Note that $X_0 \neq \emptyset$ for any $b > 0$. The following proposition describes a symmetric undominated equilibrium.

Proposition 1. Suppose that the assumption 1 holds. There exists a symmetric undominated equilibrium in which a buyer with valuations $(x_1, x_2) \in X_q$ chooses quantity q in the buyout stage and submits the single-unit bid $(p_1, p_2) = (x_1, 0)$ in the bid stage. The set of valuations X_q is given by

$$X_2 = \{(x_1, x_2) \in X | x_2 \geq h(x_1), x_2 \geq b\},$$
$$X_1 = \{(x_1, x_2) \in X | x_1 \geq \underline{x}, x_2 \leq b\},$$
$$X_0 = X/(X_2 \cup X_1),$$

where

$$h(x_1) = \frac{1}{2 - F(X_2) + 4F(X_0)}\Big(- 5F(X_2)x_1 + [8 - 2F(X_2) - F(X_0)]b\Big)$$

and

$$\underline{x} = \frac{3 - F(X_2)}{2F(X_2)}b.$$

Proof. Suppose that buyout price $b > 0$ is given. Since in the case (i) the buyer 1 optimally never chooses $q > 0$ for any buyout price $b > 0$, we consider the cases (ii) and (iii).

In the case (ii), the buyer 1 should choose $q < 2$ because the buyout price exceeds the valuation on his second unit. Suppose that $X_2 \neq \emptyset$ holds for given $b > 0$. Let $\underline{x} \geq b$ denote the valuation on the first unit of the buyer 1 at which he is indifferent between choosing $q = 1$ and $q = 0$ in the buyout stage. The valuation $\underline{x}$ should satisfy

$$[1 - \frac{1}{3}F(X_2)](\underline{x} - b) = [1 - F(X_2)]\underline{x},$$

or equivalently

$$\underline{x} = \frac{3 - F(X_2)}{2F(X_2)}b.$$

Note that $\underline{x} > b$ holds for $F(X_2) \in (0, 1)$. Since the threshold $\underline{x}$ is uniquely determined by buyout price b, the buyer 1 prefers $q = 1$ to $q = 0$ if and only if his valuation on the first unit satisfies $x_1 > \underline{x}$. On the other hand, if $X_2 = \emptyset$ for the given buyout price $b > 0$, the buyer 1 prefers $q = 0$ to $q = 1$ since $F(X_2) = 0$ yields $\Pi_1(x_1, x_2, b) = x_1 - b < \Pi_0(x_1, x_2, b) = x_1$.

In the case (iii), choosing $q = 1$ is strictly dominated by choosing $q = 2$, because $x_1 \geq x_2 > b$ yields

$$\Pi_2(x_1, x_2, b) - \Pi_1(x_1, x_2, b) = \frac{F(X_2)(x_1 - b)}{6} + [\frac{F(X_2)}{6} + \frac{F(X_1)}{3} + F(X_0)](x_2 - b) > 0.$$

Thus, let $(\hat{x}_1, \hat{x}_2)$ denote the valuations of the buyer 1 at which he is indifferent between choosing $q = 2$ and $q = 0$. The valuations should satisfy

$$[1 - \frac{1}{6}F(X_2)](\hat{x}_1 - b) + [\frac{1}{6}F(X_2) + \frac{1}{3}F(X_1) + F(X_0)](\hat{x}_2 - b) = [1 - F(X_2)]\hat{x}_1,$$

or equivalently

$$\hat{x}_2 \equiv h(\hat{x}_1) = \frac{1}{2 - F(X_2) + 4F(X_0)}\Big(- 5F(X_2)\hat{x}_1 + [8 - 2F(X_2) - F(X_0)]b\Big). \qquad (2)$$

The set of $(\hat{x}_1, \hat{x}_2)$ satisfying (2) is represented by a downward-sloping linear line in the (x_1, x_2)-plane. By a simple calculation, we obtain $h(b) = ([8 - 7F(X_2) - F(X_0)]/[2 - F(X_2) + 4F(X_0)])b > b$ and $h(\underline{x}) = ([1 + F(X_2) - 2F(X_0)]/[2(2 - F(X_2) + 4F(X_0))])b < b$. Therefore, $x_2 = h(x_1)$ intersects $x_2 = b$ at $x_1 = x^* \in (b, \underline{x})$. Specifically, $x^* = b[6 - F(X_2) - 5F(X_0)]/(5F(X_2))$.

Since these thresholds $\underline{x}$ and $\hat{x}_2 = h(\hat{x}_1)$ are uniquely determined by buyout price b and yields the triplet of sets (X_2, X_1, X_0). Thus, the buyer 1 has no incentives to deviate from choosing q based on (X_2, X_1, X_0). In the bid stage, as the lemma 1 shows, both buyers cannot profitably deviate from submitting the single-unit bid as long as the assumption 1 holds. □

Figure 1 illustrates the region of valuations that the buyer chooses q in the symmetric undominated equilibrium. The equation (2) is depicted by a downward-sloping line in the figure 1. The $x_2 = h(x_1)$

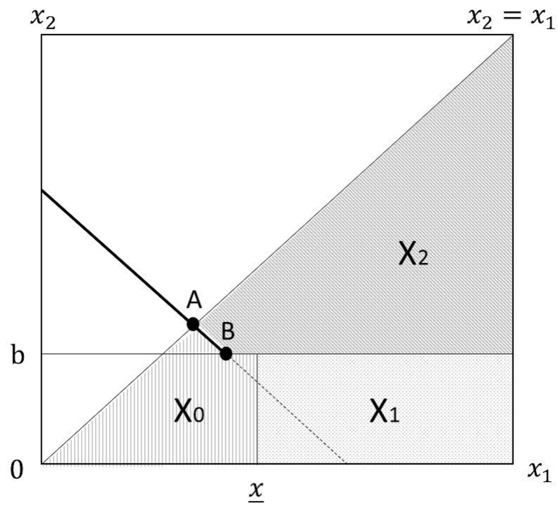

Figure 1: Thresholds

intersects $x_2 = x_1$ at the point A, which is above $x_2 = b$ because of $h(b) > b$. The point B is given by (x^*, b). Note that the buyer with higher valuations tend to exercise a buyout option for more units.

Unfortunately, it is ambiguous whether X_2 shrinks as b increases because X_q, h, and $\underline{x}$ affect each other. However, we provide the necessary condition that a buyout option is never exercised. First, we show that if a buyer chooses $q < 2$, then a buyout option is never exercised for such a buyout price.

Lemma 2. In the symmetric undominated equilibrium presented in the proposition 1, if $X_2 = \emptyset$, then $X_0 = X$.

Proof. Suppose that in the symmetric undominated equilibrium, $X_2 = \emptyset$ for some $b > 0$. We show that the buyer 1 never chooses $q = 1$ for such a buyout price. If the buyer 1 has valuations such that (i) $x_2 \leq x_1 < b$, he never chooses $q > 0$. If his valuations satisfy (ii) $x_2 < b \leq x_1$, he prefers $q = 0$ to $q = 1$ because $F(X_2) = 0$ leads to $\Pi_0(x_1, x_2, b) = x_1 > \Pi_1(x_1, x_2, b) = x_1 - b$. If his valuations satisfy (iii) $b < x_2 \leq x_1$, choosing $q = 2$ dominates $q = 1$ because $\Pi_2(x_1, x_2, b) > \Pi_1(x_1, x_2, b)$ for any $b > 0$. Thus, the buyer 1 never chooses $q = 1$. Since the equilibrium is symmetric, $X_1 = \emptyset$ should hold. Hence $X_0 = X$. □

Note that although $X_2 = \emptyset$ implies $X_1 = \emptyset$, the converse is not true. Thus, there exist three cases: (1) a buyout option is never exercised, (2) a buyer chooses $q \in \{0, 2\}$, and (3) a buyer chooses $q \in \{0, 1, 2\}$. The next proposition gives the necessary condition that a buyout option is never exercised.

Proposition 2. In the symmetric undominated equilibrium presented in the proposition 1, if $X_0 = X$ (i.e., $F(X_0) = 1$), then $b > \frac{1}{2}\bar{x}$.

Proof. Suppose that in the symmetric undominated equilibrium, $X_0 = X$ for some $b > 0$. Since $F(X_2) = F(X_1) = 0$, the buyer 1's expected payoffs are rewritten as $\Pi_2(x_1, x_2, b) = x_1 + x_2 - 2b$ and $\Pi_0(x_1, x_2, b) = x_1$. Note that $q = 1$ is strictly dominated by $q = 2$ for the buyer with valuations that satisfies $b < x_2 \leq x_1$. The buyer 1 should choose $q = 0$ if and only if $\Pi_0(x_1, x_2, b) > \Pi_2(x_1, x_2, b)$ for any (x_1, x_2). That is, $b > \frac{1}{2}\bar{x}$. □

The proposition 2 implies that the second-unit valuation plays a crucial role for a buyer to exercise a buyout option. Since seller revenue is zero without a buyout price, we immediately see the revenue-enhancing effect of a buyout price from the proposition 2.

Corollary 1. Any buyout price $b \leq \frac{1}{2}\bar{x}$ is exercised with a positive probability and increases the expected seller revenue.

5 Flat-demand example

In this section, we consider a simple case where a buyer has flat-demand (i.e., $x_1 = x_2 = x$) and his valuation is uniformly distributed in $[0, 1]$. First, we see this simple case satisfies the assumption 1. Suppose that a buyer with valuation x submits $p \in [0, 1]$ for a second unit. Since $F_1(x) = F(x) = x$ and $f_1(x) = 1$, we have

$$(x - p)\frac{f_1(p)}{1 - F_1(p)} = \frac{x - p}{1 - p} \leq 1$$

for any x and for any $p \leq x$. For the next step, we define $\bar{X} = [0, \bar{x}]$ for any $\bar{x} \in [0, 1]$. Since $F_1(x|\bar{x}) = \frac{x}{\bar{x}}$ and $f_1(x|\bar{x}) = \frac{1}{\bar{x}}$, we have

$$(x - p)\frac{f_1(p|\bar{x})}{1 - F_1(p|\bar{x})} = \frac{x - p}{\bar{x} - p} \leq 1$$

for any $x \in \bar{X} = [0, \bar{x}]$ and for any $p \leq x$. Therefore, by the lemma 1, the single-bid in the bid stage constitutes a symmetric undominated equilibrium.

Since the lemma 1 applies to this case as well, we should consider two thresholds $\bar{c}$ and $\underline{c}$. First, a buyer with a valuation at or above threshold $\bar{c}$ exercises a buyout option for two units. Second, a buyer with a valuation below the threshold $\bar{c}$ but at or above threshold $\underline{c}$ exercises a buyout option for one unit. Finally, a buyer with a valuation below the threshold $\underline{c}$ does not exercise a buyout option in the bid stage. Since a buyer with a valuation below buyout price b should never exercise a buyout option, the threshold $\underline{c}$ should be at or below the buyout price, i.e., $\underline{c} \geq b$ We consider an equilibrium consisted of these thresholds $(\bar{c}, \underline{c})$ where $\bar{c} \geq \underline{c}$ holds.

The expected payoffs can be written as follows.

$$\begin{aligned}
\Pi_2(x,b) &= (1-\bar{c})[\frac{1}{6}(2x-2b)+\frac{4}{6}(x-b)] + (\bar{c}-\underline{c})[\frac{1}{3}(2x-2b)+\frac{2}{3}(x-b)] + \underline{c}(2x-2b) \\
&= (1+\frac{1}{3}\bar{c}+\frac{2}{3}\underline{c})(x-b), \\
\Pi_1(x,b) &= (1-\bar{c})\frac{2}{3}(x-b) + \bar{c}(x-b) \\
&= (\frac{2}{3}+\frac{1}{3}\bar{c})(x-b), \\
\Pi_0(x,b) &= \bar{c}x.
\end{aligned}$$

For a buyer with valuation $x \geq b$, choosing $q = 1$ is strictly dominated by choosing $q = 2$ since $\Pi_2(x,b) > \Pi_1(x,b)$ holds; thus, we suffice to compare two choices $q = 2$ and $q = 0$. Let c be a threshold that a buyer with a valuation at or above threshold c exercises a buyout option for two units. The expected payoffs can be rewritten as follows.

$$\begin{aligned}
\Pi_2(x,b) &= (1-c)[\frac{1}{6}(2x-2b)+\frac{4}{6}(x-b)] + c(2x-2b) = (1+c)(x-b), \\
\Pi_0(x,b) &= cx.
\end{aligned}$$

Since a buyer should be indifferent between choosing $q = 2$ and $q = 0$ if his valuation is c, we obtain

$$(1+c)(c-b) = c^2,$$

or equivalently $c = b/(1-b)$. Figure 2 illustrates the relation between buyout price b and threshold c. The horizontal and vertical axises represent b and c, respectively. As shown in the figure 2, threshold c increases in buyout price b, and a buyout price is never exercised if $b > 1/2$ because $c > 1$ holds for $b > 1/2$. The seller revenue corresponding buyout price b is given by

$$U(b) = \Big[1 - (\frac{b}{1-b})^2\Big]2b = \frac{2(1-2b)b}{(1-b)^2}.$$

Thus, the first-order condition yields the optimal buyout price $b^* = 1/3$, leading to $U(1/3) = 1/2$. Figure 3 illustrates the seller revenue. In the figure 3, the horizontal and vertical axises represent b and U, respectively. The dotted line represents $U(b)$. The buyout option weakly improves seller revenue since the single-unit bid yields a zero revenue. Note that a sincere bid on both units yield a higher seller revenue (and perfect efficiency) which is depicted by a solid line in the figure. If buyers are forced to submit a flat-demand, a dominant strategy for them, increasing in seller revenue. To exercise a buyout option has a perspective of a flat-bid; thus, seller revenue increases with the introduction of a buyout option.

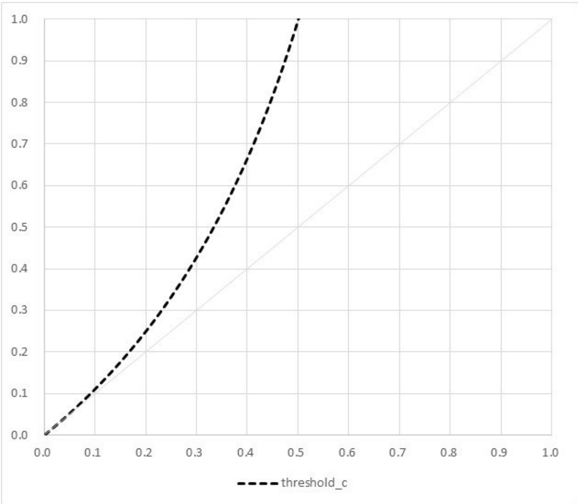

Figure 2: Threshold

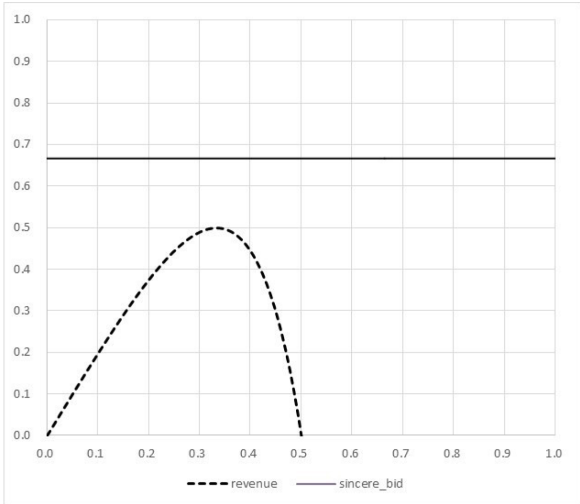

Figure 3: Expected seller revenue

6 Concluding remarks

This paper studies the impact of a buyout price on strategies of buyers and seller revenue in uniform-price sealed-bid auctions and suggest the introduction of a buyout price improves seller revenue by deterring a demand reduction. As well known, seller revenue tends to be low in uniform-price auctions because buyers lower a bid on a second unit. This phenomenon is known as a demand reduction, and as the extreme case, theoretically, buyers submit a zero price for a second unit, yielding a zero revenue to a seller. A demand reduction is widely observed in the experimental and real world multi-unit auctions, and thus the low revenue caused by a demand reduction matters. This paper suggests that a buyout price solves this issue. Multi-unit auctions have grown increasingly important in terms of a practical usage, and thus the result obtained in this paper is important.

A buyout option can perform better in multi-unit auctions than in single-unit ones. Chen, et al. (2013) study a buyout price in second-price sealed-bid auctions where a risk-averse buyer has single-unit demand and show that a buyout price improves seller revenue. They discuss that a seller can be seriously harmed by posting an inappropriate buyout price. This case happens if a seller does not know the distribution of valuation or the degree of risk aversion. This does not matter in multi-unit auctions when low revenues indeed realize. In this sense, the introduction of a buyout price is less "risky" in multi-unit uniform-price auctions as compared with single-unit auctions.

We consider a simple situation in this paper, and thus the analysis has a clear limitation. First, we restrict our attention to the case where buyers bid zero on a second unit. However, the revenue-enhancing effect of buyout prices do not necessarily rely on a single-unit bid. A buyout price creates a trade-off faced to buyers as long as a demand reduction emerges, although the revenue-enhancing effect is moderated.

Second, we focus on the situation where only two buyers are involved. It seems that a buyout price can improve seller revenue as long as a demand reduction occurs even though more buyers participate auctions. Engelbrecht-Wiggans, et al. (2006) conduct a field experiment where three or five bidders with two-unit demand compete in bids for sports trading cards in uniform-price and Vickrey sealed-bid auctions, and compare their results with those of List and Lucking-Reiley (2000) who consider a two-bidder case. They find that in most cases, bidders on average submit lower second-unit bids in the uniform-price auctions than in the Vickrey auctions. Moreover, a change of the number of bidders from three to five increases the second-unit bid in the uniform-price sealed-bid auctions. However, they report that these differences are statistically insignificant.

Third, we focus on buyers demanding at most two units of items. As List and Lucking-Reiley (2000) note that an increase in the number of items intensifies a demand reduction. Thus, a buyout price might play a more effective role to prevent seller revenue from suffering from a demand reduction.

We close this paper by directing the future research. First, as we stated above, a two-buyer two-item case is just a first step. One can naturally extend our analysis to a general case where $k \geq 2$ items are sold against $n \geq 2$ buyers. The difficulty arises at considering the buyout stage because calculating a winning probability to obtain a certain unit becomes complicated.

Second, we can conduct an experiment and test the revenue-enhancing effect of a buyout price in uniform-price sealed-bid auctions. The experimental design of single-unit auctions with a buyout price (e.g., Ivanova-Stenzel and Kroger, 2008) can directly apply to multi-unit auctions with a buyout price.

Third, it seems important to consider the impact of a buyout price on seller revenue in multi-unit ascending auctions because the experimental studies report a more considerable level of demand reductions (and hence low seller revenue) in ascending auction (Alsemgeest, et al., 1998; Porter and Vragov, 2006; Engelmann and Grimm, 2009). Moreover, the FCC spectrum and the second-generation spectrum auctions that present a demand reduction are multi-unit ascending auctions. We await the further understanding of multi-unit auctions and a buyout option from future researches.

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