Trade-off of Equilibrium Refinement: An Example of First-price Auctions with Uncertain Number of Bidders

Toshihiro Tsuchihashi[(1) (2)]

Abstract. It is common to employ a certain criterion for equilibrium refinement in the literature on signaling games. Specifically, a scope is focused on whether there exists the unique equilibrium outcome that survives criterions. However, there seems no consensus which criterion to employ for refinement. Then, it is not a good idea to restrict our attentions on the unique equilibrium outcome with some favored criterion. Instead, we had better look at an entire set of equilibriums. This short paper characterizes an entire set of equilibrium outcomes in a first-price auction signaling game and discusses that costs for refinement could dominate a benefit from refinement.

Keywords. Signaling game; First-price auction; Refinement; Entire set of equilibriums

I. Introduction

A continuum of equilibriums exists in signaling games, but generally researchers are under pressure to show uniqueness of equilibriums. It is common to employ a certain criterion for equilibrium refinement in the literature on signaling games. Specifically, researchers focus on whether *there exists the unique equilibrium outcome that survives some criterions*. The examples of such criterions include Intuitive Criterion (Cho and Kreps, 1987), D1 (Banks and Sobel, 1987), Never Weak Best Response (Kohlberg and Mertens, 1986), and Universal Divinity (Banks and Sobel, 1987). In the sender-receiver game, Cho and Sobel (1990) argue that an equilibrium outcome is less likely to survive Universal Divinity than D1.

This trend for refinement is fair in the sense that the unique equilibrium provides better predictions and guidance of agents' behaviors. Theoretically, it is true.

---

(1) Faculty of Economics, Daito Bunka University, 1-9-1 Takashimadaira, Itabashi, Tokyo, 175-8571 Japan: tsuchihasi@ic.daito.ac.jp

(2) This research is based on the analysis of Tsuchihashi (2016). I am truly indebted to the attendees at the Lunch Time Seminar in Daito Bunka University for their valuable comments and suggestions to Tsuchihashi (2016). All remaining errors are of course my own responsibilities.

Apart from theory, what criterion for refinements shows better performance in experiments? The results vary across the research (Banks, Camerer, and Porter, 1994; Brandts and Holt, 1992). Moreover, "contest effects" can matter for equilibrium selection in signaling games (Cooper and Kagel, 2003), even though this is not surprising to psychologists. There seems no consensus which criterion to employ for refinement.

These chaotic situations observed in the literature may suggest that it is not a good idea to restrict our attentions on the unique equilibrium outcome with some favored criterion. Instead, we had better look at an entire set of equilibriums. For signaling games under consideration, the best thing we can do is to provide all possibilities to happen, even though it seems less attractive than the unique prediction on what happens.

Consequently, in this short paper, I characterize a whole set of equilibrium outcomes in first-price auctions in which either one or two bidders participate(s) and the number of bidders is a seller's private information. With the intention to characterize a whole set of equilibriums, the current paper assumes the most pessimistic off-the-equilibrium-path belief, that is, a bidder believes to be alone in the auction when observing any reserve prices off the equilibrium path. Moreover, for the simplicity, I assume the uniform distribution on bidder valuations. This auction model shares some flavors with Tsuchihashi's (2016) setting.

The current research extends the analysis of Tsuchihashi (2016) in a significant direction, that is, my previous paper focuses only on a separating equilibrium whereas the current research aims at finding a whole set of equilibrium outcomes. Further, there is another difference between the two papers that the potential number of bidders is two or three in the previous research.

In what follows, I describe my setup in Section 2. Sections 3 and 4 characterize separating and pooling equilibriums, respectively. In Section 5, I address the result and conclusion.

II. Preliminary

Either one or two bidder(s) participate(s) in a first-price auction. The number of active bidders is a seller's private information, but a probability that only one bidder is active, p, is common knowledge. The bidder's private valuation, x, is a random draw from $[0,1]$ with the uniform distribution function $F(x) = x$. For simplicity, the seller's valuation is normalized to zero.

The auction is the following two-stage game. First, the seller observes the number of active bidders, n, and then chooses a reserve price, r. Second, the bidder forms a belief on the number of bidders, q, and then submits a bid.

I employ the Perfect Bayesian equilibrium (PBE) as a solution concept. In order to obtain

PBEs, I assume the most pessimistic belief. A bidder believes that a seller type is $n = 1$ with probability one when observing off-the-equilibrium reserve prices. This auction game is a standard signaling game: The number of active bidders is the seller's type. Type n seller chooses a reserve price in order to maximize the expected revenue $U_n(r, q)$. For convenience, I let $q = 1$ and $q = 2$ denote the beliefs that "one bidder participates," and that "two bidders participate," respectively. Further, with a little abuse of notation, I write $q = p$ when the ex-post belief is equivalent to the ex-ante belief.

III. Separating equilibrium

First, I provide type 1 seller's optimal reserve price. The least type seller, facing one bidder, optimally chooses the reserve price as if the number of bidders is common knowledge. Intuitively, bidders assign probability one to type 1 seller for any reserve prices but the on-the-equilibrium-path reserve price type 2 seller chooses. A bidder's optimal bidding function is $b_1(x, r) = r$ when he believes that he is an only bidder, then, type 1 seller chooses $r_1 = 1/2$ because she maximizes

$$U_1(r, 1) = r[1 - F(r)] = r(1 - r).$$

Then, I write the equilibrium payoff of type 1 as $\widetilde{U}_1 = 1/4$.

Second, I provide type 2 seller's optimal reserve price. With a standard calculation appearing in textbooks, the bidder with belief $q = 2$ submits a bid according to $b_2(x, r) = (x^2 + r^2)/2x$. Thus, given beliefs $q = 1$ and $q = 2$, type 2 seller obtains the expected payoffs

$$U_2(r, 1) = r[1 - F(r)^2] = r(1 - r^2),$$

and

$$U_2(r, 2) = 2 \times \int_r^1 b_2(x, r) f(x) dx = \frac{1 - r^2(1 + \log r^2)}{2}.$$

In equilibrium, r_2 should satisfy two conditions. The first condition is type 1 seller's incentive compatibility:

$$U_1(r_1, 1) \geq U_1(r_2, 2) = \int_{r_2}^1 b_2(x, r_2) f(x) dx,$$

or equivalently,

$$r_2 \geq e^{-\frac{1}{2}}.$$

Figure 1 illustrates this condition. In the figure, the horizontal and vertical axes are reserve price r_2 and type 1 seller's expected payoff, respectively. The dashed line represents the expected payoff of type 1 seller to deviate to r_2.

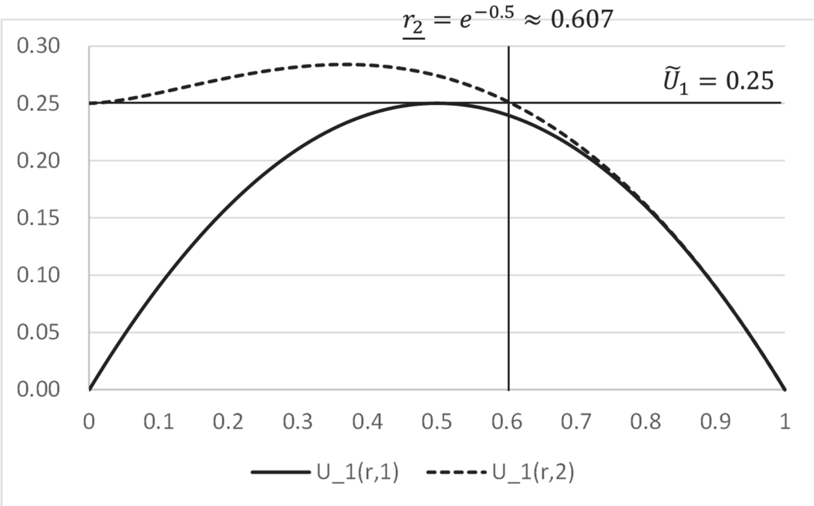

Figure 1. Type 1 seller's incentive compatibility condition

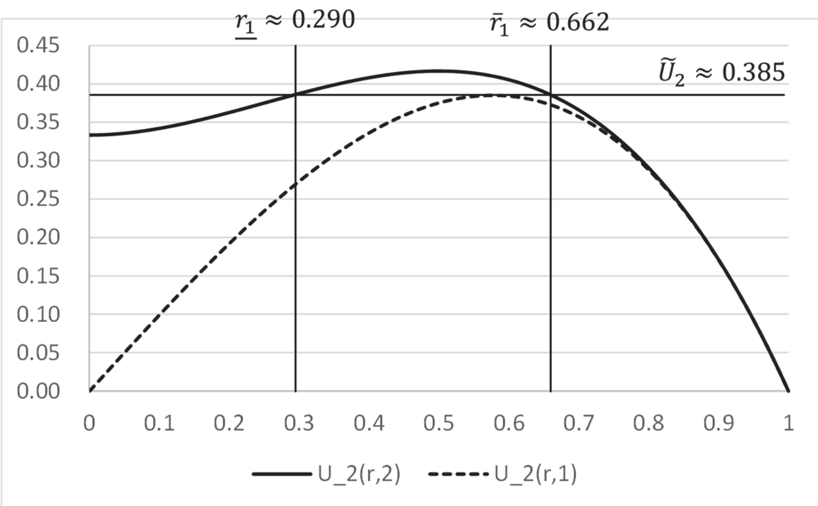

Figure 2. Type 2 seller's incentive compatibility condition

The second condition is type 2 seller's incentive compatibility. Note that the condition is not $U_2(r_2, 2) \geq U_2(r_1, 1)$. Instead, one can consider the most profitable deviation for type 2 seller, which is $\tilde{r} = 3^{-1/2}$ because $\tilde{r} = \arg\max U_2(r,1)$. I let $\widetilde{U}_2$ denote the maximum of type 2 seller's expected deviation payoff. Thus, type 2 seller's incentive compatibility is

$$U_2(r_2, 2) \geq U_2(\tilde{r}, 1) = \widetilde{U}_2,$$

or equivalently,

$$0.290 \approx \underline{r_1} \leq r_2 \leq \bar{r}_1 \approx 0.662.$$

Figure 2 illustrates this condition. In the figure, the horizontal and vertical axes are reserve price r_2 and type 2 seller's expected payoff, respectively.

Therefore, I obtain the following proposition that characterizes a set of all separating equilibrium outcomes.

Proposition 1. *Accompanied by the most pessimistic belief, the reserve prices* $r_1 = 1/2$ *and* $\underline{r_2} \leq r_2 \leq \bar{r}_1$ *that satisfies* $U_2\left(\underline{r_1}, 2\right) = U_2(\bar{r}_1, 2) = U_2(\tilde{r}, 1)$ *constitute a separating PBE.*

Clearly, type 1 seller's choice is optimal in the sense that she chooses a reserve price as if her type is public information. On the other hand, type 2 seller is forced to sub-optimally increase a reserve price. Note that it is never optimal for type 2 to choose $r < 1/2$, that is, equilibrium reserve prices should increase with a seller type.

IV. Pooling equilibrium

Let r_p be an equilibrium reserve price in a pooling PBE. A bidder with the ex-ante belief submits a weighted-average bid, which is given by

$$b_p(x, r) = \frac{p}{p + (1-p)x} b_1(x, r) + \frac{(1-p)x}{p + (1-p)x} b_2(x, r).$$

Note that the weight depends on the bidder's valuation. See Harstad et al. (1990) and Tsuchihashi (forthcoming) for the derivation of the weighted-average bid.

First, I consider type 1 seller's equilibrium decision. By choosing reserve price r, Type 1 seller obtains the expected payoff of

$$U_1(r, p) = \int_r^1 b_p(x, r) f(x) dx.$$

The expected payoff should be the weighted average of $U_2(r,1)$ and $U_2(r,2)$ because the bidding function $b_p(x,r)$ is the weighted average of $b_1(x,r)$ and $b_2(x,r)$, even though these weights can be different.

The most profitable deviation for type 1 seller is the same as above, that is, she would obtain $U_1(0.5,1)$ by choosing $r = 0.5$ off the equilibrium path. Thus, Figure 1 suggests that a range of equilibrium pooling reserve prices should be wider than the one in separating equilibriums.

Second, I consider type 2 seller's equilibrium decision. By choosing reserve price r, Type 2 seller obtains the expected payoff of

$$U_2(r,p) = \int_r^1 b_p(x,r)\, f_2(x)dx.$$

The most profitable deviation for type 2 seller is also the same as above, that is, she would obtain $U_2(\tilde{r},1)$ by choosing $\tilde{r}$ off the equilibrium path. Thus, Figure 2 suggests that a range of equilibrium pooling reserve prices should be narrower than the one in separating equilibriums.

Therefore, I obtain the following proposition that shows a set of all pooling equilibrium outcomes.

Proposition 2. *Let r_p be an equilibrium reserve price in pooling equilibriums. Then, accompanied by the most pessimistic belief, the reserve price should lie in $\left(\underline{r_p}, \bar{r}_p\right)$ that satisfies*

$\left(\underline{r_p}, \bar{r}_p\right) \subset \left(\underline{r_2}, \bar{r}_1\right)$.

Proposition 2 implies that the first-best outcome cannot realize since $\underline{r_2} > 0.5$. Interestingly, type 1 seller is worse off in pooling equilibriums whereas it is case-bay-case which equilibrium yields the higher expected payoff to type 2 seller. Thus, a pooling equilibrium is never Pareto optimal.

V. Result and conclusion

In this short paper, I assumed the most pessimistic belief and derived an entire set of equilibrium outcomes. Propositions 1 and 3 show an entire set of equilibriums, which is illustrated in Figure 3. Intuitively, the set seems very "small" in the two-dimension profile space, that is, the equilibrium reserve prices lie in narrow ranges.

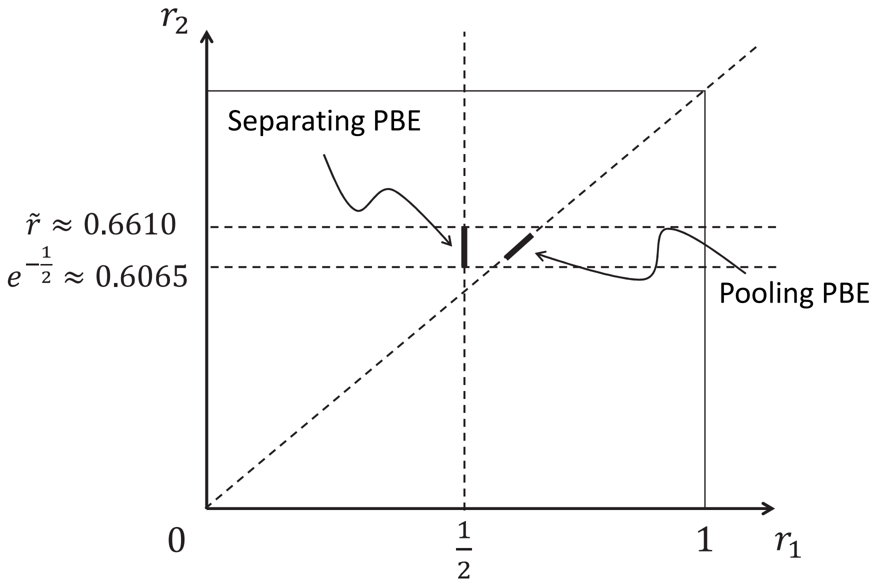

Figure 3. Separating and pooling PBEs

One would expect the uniqueness result by employing, say, D1 under the current paper's setup. A profile of $(r_1, r_2) = (0.5, \overline{r_1})$ survives D1. This may be the unique D1 outcome because the current paper's auction model satisfies Cho and Sobel's (1990) sufficient condition for uniqueness. Note, however, that Cho and Sobel (1990) cannot directly apply to the current paper (See Section 6.1 in Tsuchihashi (2020) for the detail).

Refinement is costly. The analysis requires an author to make a cumbersome calculation and readers face to difficulty for understanding the paper. Clearly, there are benefits from refinement: the analysis provides better predictions and guidance of agents' behaviors. The current research may cast a doubt on a general trend where an equilibrium *should* be unique and mathematically complicated concepts *should* serve for that purpose.

References

Banks, Jeffrey, Colin Camerer, and David Porter, "An Experimental Analysis of Nash Refinements in Signaling Games," *Games and Economic Behavior*, Volume 6, Issue 1, January 1994, pp. 1–31.

Banks, Jeffrey S. and Joel Sobel, "Equilibrium Selection in Signaling Games," *Econometrica*, Volume 55, No. 3, May 1987, pp. 647–661.

Cho, In-Koo, and David M. Kreps, "Signaling Games and Stable Equilibria," *The Quarterly Journal of Economics*, Volume 102, Issue 2, May 1987, pp. 179–221.

Cho, In-Koo and Joel Sobel, "Strategic Stability and Uniqueness in Signaling Games," *Journal of Economic Theory*, Volume 50, Issue 2, April 1990, pp. 381–413.

Brandts, Jordi and Charles A. Holt, "An Experimental Test of Equilibrium Dominance in Signaling Games," *The American Economic Review*, Volume 82, No. 5, December 1992, pp. 1350–1365.

Cooper, David J and John H. Kagel, "The Impact of Meaningful Context on Strategic Play in Signaling Games," *Journal of Economic Behavior & Organization*, Volume 50, Issue 3, March 2003, pp. 311–337.

Harstad, Ronald M., John H. Kagel, and Dan Levin, "Equilibrium Bid Functions for Auctions with an Uncertain Number of Bidders," *Economics Letters*, Volume 33, Issue 1, May 1990, pp. 35-40.

Kohlberg, Elon and Jean-Francois Mertens, "On the Strategic Stability of Equilibria," *Econometrica*, Volume 54, Issue 5, September 1986, pp. 1003—1037.

Tsuchihashi, Toshihiro, "Auctions with Uncertain Number of Bidders with Reserve Prices," Daito Bunka University Discussion Paper No. 16-3, 2016.

Tsuchihashi, Toshihiro, "Reserve Price Signaling in First-price Auctions with an Uncertain Number of Bidders," *International Journal of Game Theory,* Volume 49, Issue 4, December 2020, pp. 1081-1103.